Topic 6 -

Vector Spaces

HW 6 Topic - Vector Spaces

We are going to generalize what a scalar/number is.] Field

Then we will generalize what a vector is.] vector space

Def: A field consists of a set F of "scalars" or "numbers" and two operations $+$ and $\cdot$ such that if x and y are scalars in F then there exists unique elements $x+y$ and $x \cdot y$ in F.
Also the following properties must hold:

(F1) If a, b, c are in F, then:

$$a+b=b+a$$
$$a \cdot b = b \cdot a$$
$$a+(b+c)=(a+b)+c$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$
$$a \cdot (b+c) = a \cdot b + a \cdot c$$
$$(b+c) \cdot a = b \cdot a + c \cdot a$$

(F2) There exist unique elements 0 and 1 in F where

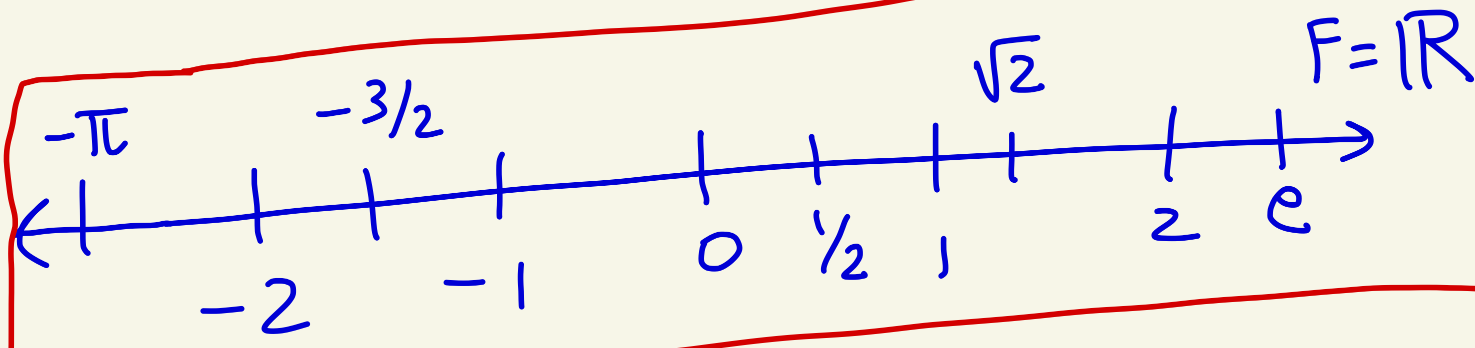
$$x+0=0+x=x \quad \text{and} \quad 1 \cdot x = x \cdot 1 = x$$

for all x in F.

(F3) Let x be in F.
Then there exists a unique element $-x$ in F where $x+(-x)=0$ and $(-x)+x=0$.
In addition if $x \neq 0$, then there exists a unique element x^{-1} in F where

$$x \cdot (x^{-1}) = 1 \quad \text{and} \quad (x^{-1}) \cdot x = 1.$$

Ex: $F = \mathbb{R}$, the set of real numbers, is a field using the usual $+$ and $\cdot$.



Why is $\mathbb{R}$ a field?

- Adding and multiplying real numbers gives a real number.

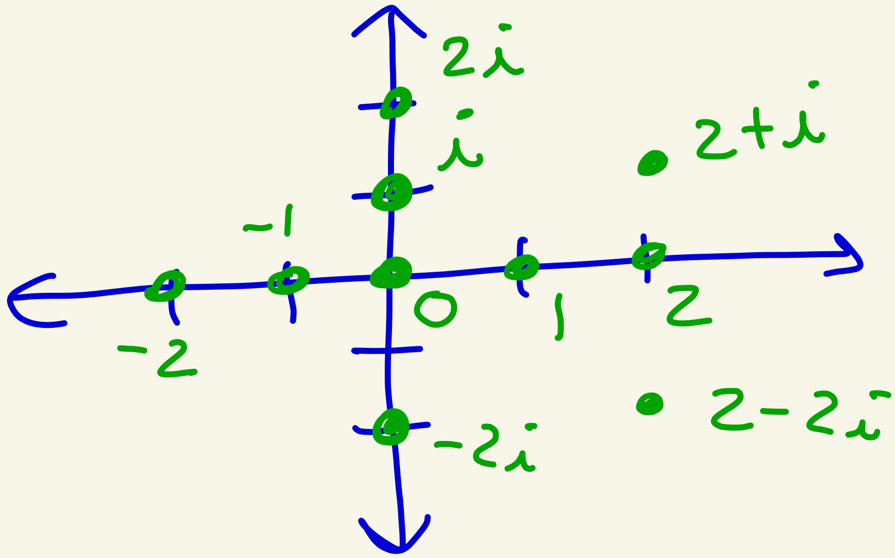
(1) All the properties from (F1) are true in $\mathbb{R}$.

(2) $\mathbb{R}$ has elements 0 and 1 that behave as in (F2).

(3) We have (F3) is true.

Note: In our class, $\mathbb{R}$ is the only field that we will use. But let's see some others just to see.

<u>Ex:</u> The set of complex numbers $\mathbb{C}$ is a field.



We won't use this field in this class.

<u>Ex:</u> There even exist fields that are finite in size. You get these by "modular arithmetic".

For our class, we will always use $\mathbb{R}$ as our field.

But I will state theorems for general fields.

Now we generalize what a "vector" is.

Def: A *vector space* V over a field F consicts of a set of "vectors" V and a field F with two operations, "vector addition" $+$ and "vector scaling" $\bullet$, such that if $\vec{v}, \vec{w}, \vec{z}$ are vectors in V and α, β are scalars from F then the following must hold:

1. $\vec{v} + \vec{w}$ is in V.
2. $\alpha \cdot \vec{v}$ is in V
3. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
4. $\vec{v} + (\vec{w} + \vec{z}) = (\vec{v} + \vec{w}) + \vec{z}$
5. there exists a unique vector $\vec{0}$ in V such that $\vec{0} + \vec{y} = \vec{y} + \vec{0} = \vec{y}$ for any $\vec{y}$ in V.
6. there exists a vector $-\vec{v}$ in V where $\vec{v} + (-\vec{v}) = \vec{0}$ and $(-\vec{v}) + \vec{v} = \vec{0}$
7. $1 \cdot \vec{v} = \vec{v}$
8. $(\alpha\beta) \cdot \vec{v} = \alpha \cdot (\beta \cdot \vec{v})$
9. $\alpha \cdot (\vec{v} + \vec{w}) = \alpha \cdot \vec{v} + \alpha \cdot \vec{w}$
10. $(\alpha + \beta) \cdot \vec{v} = \alpha \cdot \vec{v} + \beta \cdot \vec{v}$

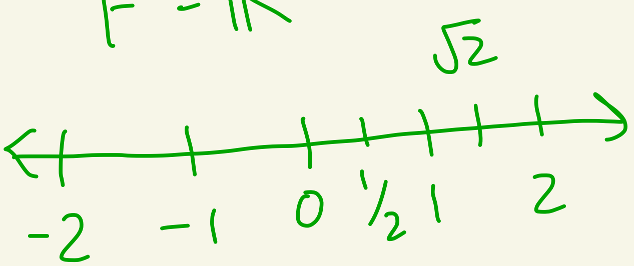
$\boxed{\text{Ex:}}$ Let $V=\mathbb{R}^n$ and $F=\mathbb{R}$

$\mathbb{R}^n$ is a vector space over the field $\mathbb{R}$ using the usual vector addition and scalar multiplication.

$\boxed{\text{Ex:}}$ $\boxed{n=2}$

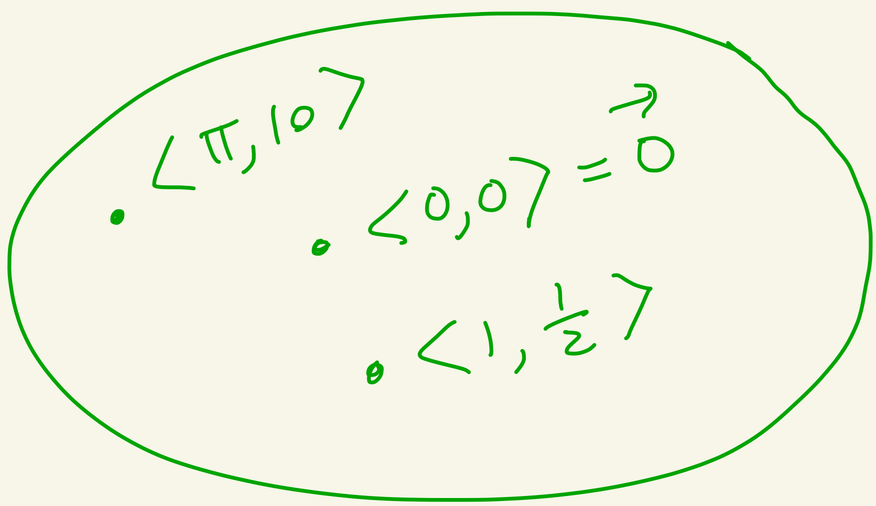
<u>Field</u>

$F=\mathbb{R}$



<u>Vectors</u>

$V=\mathbb{R}^2$



<u>vector addition:</u>

$$\langle 1, \tfrac{1}{2}\rangle + \langle 0, -5\rangle = \langle 1, -\tfrac{9}{2}\rangle$$

<u>scalar multiplication:</u>

$$5\cdot\langle 1, -2\rangle = \langle 5, -10\rangle$$

One can check that this example satisfies all 10 properties of being a vector space. Some we did in class and HW in earlier topics.

Ex: Let

$$V = M_{2,2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \text{ are real numbers} \right\}$$

$$= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 5 & \pi \end{pmatrix}, \begin{pmatrix} \sqrt{2} & \frac{1}{2} \\ 5 & 3 \end{pmatrix}, \ldots \right\}$$

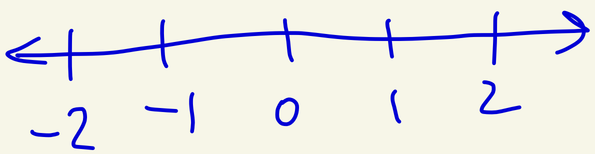
infinitely many more

these are the "vectors"

and $F = \mathbb{R}$

scalars

field $F = \mathbb{R}$



Vectors $V = M_{2,2}$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ • $\begin{pmatrix} 5 & -1 \\ \frac{1}{2} & 0 \end{pmatrix}$ • $\begin{pmatrix} \pi & \pi \\ \pi & \pi \end{pmatrix}$ •

We will use the usual addition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

and scalar multiplication

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

One can check that the 10 vector space properties hold.

Here the zero vector is

$$\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and the additive inverse of $\vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $-\vec{v} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

So, $V = M_{2,2}$ is a vector space over the field $F = \mathbb{R}$.

Ex: Pick some integer $n \geq 0$

(So, n can be $0, 1, 2, 3, 4, \ldots$)

Let V be the set of all polynomials of degree $\leq n$, denoted by P_n.

So,

$$V = P_n = \left\{ \underbrace{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n}_{\text{"vectors"}} \;\middle|\; \begin{array}{l} a_0, a_1, \ldots, a_n \\ \text{are real numbers} \end{array} \right\}$$

Let $F = \mathbb{R}$. ← scalars

Define vector addition as the usual polynomial addition

That is,

$$(a_0+a_1x+\dots+a_nx^n)+(b_0+b_1x+\dots+b_nx^n)$$
$$=(a_0+b_0)+(a_1+b_1)x+\dots+(a_n+b_n)x^n$$

Scalar multiplication is

$$\alpha(a_0+a_1x+\dots+a_nx^n)$$
$$=(\alpha a_0)+(\alpha a_1)x+\dots+(\alpha a_n)x^n$$

Two polynomials are defined to be equal if they have the same coefficients. That is,

$$a_0+a_1x+\dots+a_nx^n=b_0+b_1x+\dots+b_nx^n$$

if and only if

$$a_0=b_0,\ a_1=b_1,\ \dots,\ a_n=b_n$$

Here,

$$\vec{0} = 0 + 0x + 0x^2 + \dots + 0x^n$$

and

$$-(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n)$$
$$= (-a_0) + (-a_1)x + (-a_2)x^2 + \dots + (-a_n)x^n$$

One can verify that properties 1 - 10 are true and hence $V = P_n$ is a vector space over $F = \mathbb{R}$.

Ex: $F = \mathbb{R}$ ← scalars

$V = P_3$ ← vectors

$$= \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \;\middle|\; \begin{array}{l} a_0, a_1, a_2, a_3 \\ \text{are in } \mathbb{R} \end{array} \right\}$$

$$= \left\{ \underbrace{1 - x}_{1+(-1)x+0x^2+0x^3},\ \underbrace{\vec{0}}_{0+0x+0x^2+0x^3},\ \underbrace{10}_{10+0x+0x^2+0x^3},\ 1 + x - x^2 + x^3,\ \underbrace{x^3}_{0+0x+0x^2+1\cdot x^3},\ \underbrace{\cdots}_{\text{infinitely many more}} \right\}$$

<u>Examples of adding & scaling:</u>

$$(1-x) + (1+x-x^2+x^3) = 2 - x^2 + x^3$$

$$5(1+x-x^2+x^3) = 5 + 5x - 5x^2 + 5x^3$$

Notice that P_3 behaves like $\mathbb{R}^4$. The powers of x are like placeholders.

$$(1+2x-x^2+x^3)+(5-x+x^2+2x^3)$$

$$= 6+x \qquad +3x^3$$

This is like

$$\langle 1,2,-1,1\rangle + \langle 5,-1,1,2\rangle$$

$$= \langle 6,1,0,3\rangle$$

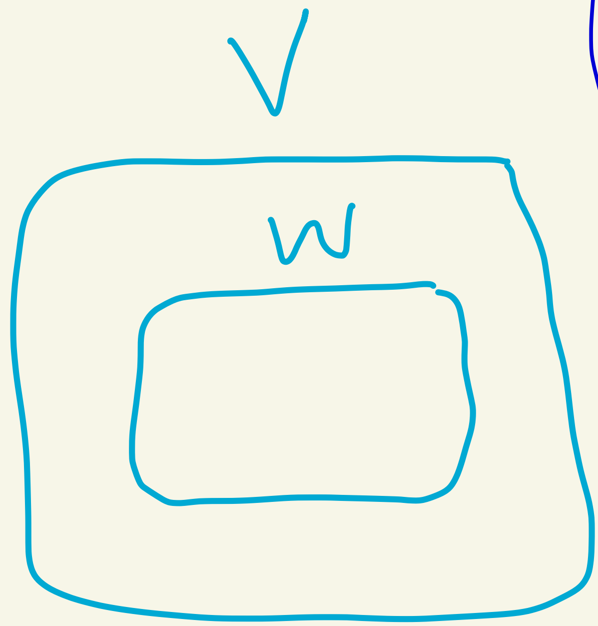
and scaling

$$3\cdot(1+x-x^2+5x^3) = 3+3x-3x^2+15x^3$$

thats like

$$3\langle 1,1,-1,5\rangle = \langle 3,3,-3,15\rangle$$

Def: Let V be a vector space over a field F. Let W be a subset of V. We say that W is a <u>subspace</u> of V if the following three conditions hold:

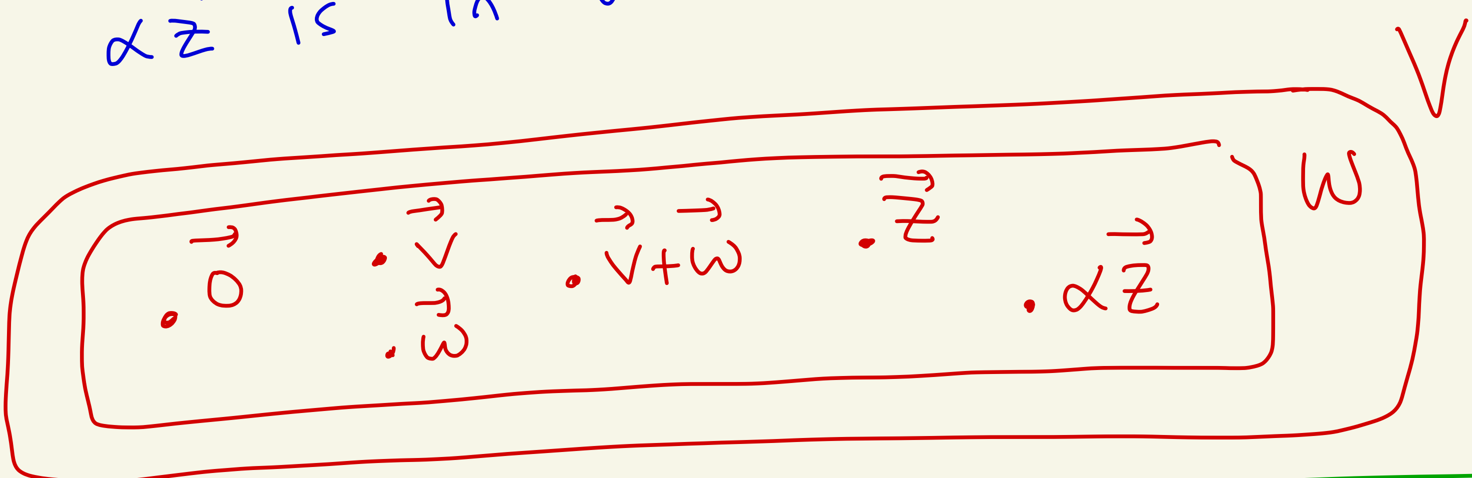


① $\vec{0}$ is in W.

② If $\vec{v}$ and $\vec{w}$ are in W, then $\vec{v}+\vec{w}$ is in W.

③ If $\vec{z}$ is in W and α is in F, then $\alpha\vec{z}$ is in W.

(① and ②: W is closed under vector addition. ③: W is closed under scaler multiplication.)



<u>Note:</u> One can show that if W is a subspace of V if and only if W itself is a vector space living inside of V.

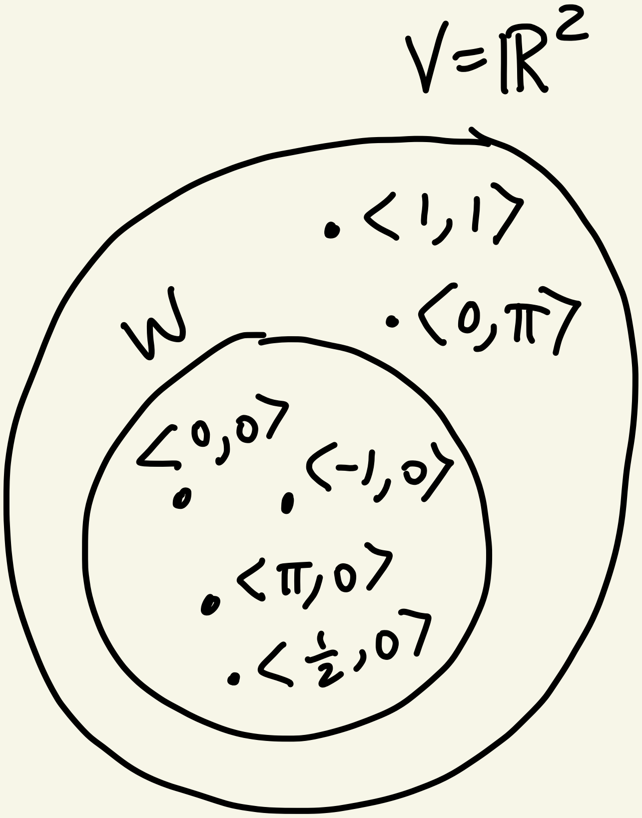
Ex: Consider the vector space $V = \mathbb{R}^2$ over the field $F = \mathbb{R}$.

Let

$$W = \{ \langle x, 0 \rangle \mid x \in \mathbb{R} \}$$
$$= \{ \langle 0,0 \rangle, \langle -1, 0 \rangle, \langle \pi, 0 \rangle, \ldots \}$$

$x = 0$ $\quad$ $x = -1$ $\quad$ $x = \pi$

infinitely many more



Let's prove that W is a subspace of V.

proof:

① Set $x=0$ in $\langle x,0\rangle$ and we get that $\langle 0,0\rangle = \vec{0}$ is in W.

② Let $\vec{v}, \vec{w}$ be in W.
Then, $\vec{v} = \langle x_1, 0\rangle$ and $\vec{w} = \langle x_2, 0\rangle$
where $x_1, x_2 \in \mathbb{R}$.
Then, $\vec{v} + \vec{w} = \langle x_1 + x_2, 0\rangle$
which is an element of W.

③ Let $\vec{z}$ be in W and α be in $F = \mathbb{R}$.
Since $\vec{z}$ is in W we know that
$\vec{z} = \langle x, 0\rangle$ where $x \in \mathbb{R}$.
Then, $\alpha\vec{z} = \alpha\langle x, 0\rangle = \langle \alpha x, 0\rangle$
which is an element of W.

By ①, ②, and ③ we have that W is a subspace of $V = \mathbb{R}^2$ ▨

Ex: Consider the vector space $V=\mathbb{R}^2$ over $F=\mathbb{R}$.

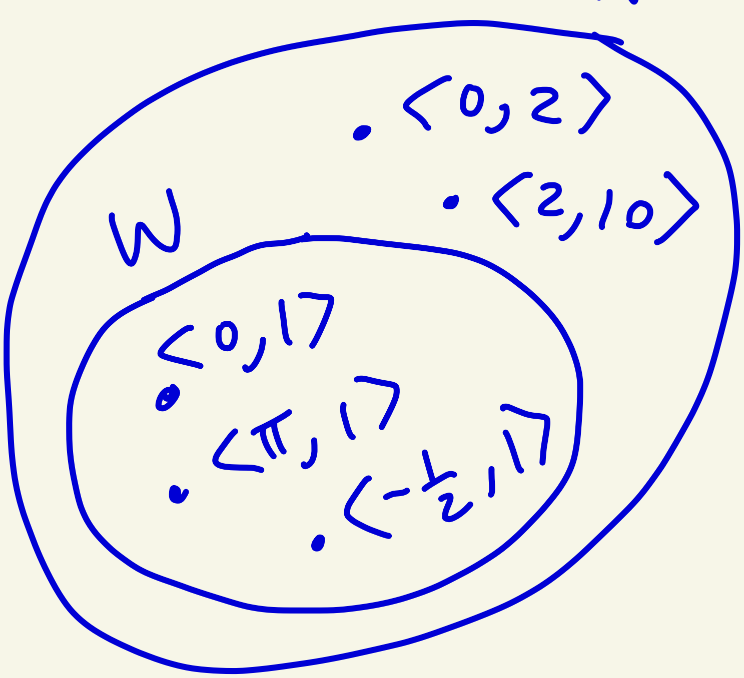
Consider

$$W = \{ \langle x, 1\rangle \mid x \in \mathbb{R} \}$$

$$= \{ \underbrace{\langle 0, 1\rangle}_{x=0}, \underbrace{\langle \pi, 1\rangle}_{x=\pi}, \underbrace{\langle -\tfrac{1}{2}, 1\rangle}_{x=-\frac{1}{2}}, \dots \}$$

infinitely many more

$V=\mathbb{R}^2$



It turns out that W is <u>not</u> a subspace of $V=\mathbb{R}^2$.

<u>For example:</u>

① Note that $\vec{0}=\langle 0,0\rangle$ is not of the form $\langle x,1\rangle$. Thus, $\vec{0}\notin W$. So W is not a subspace of $V=\mathbb{R}^2$.

One could also show that ② or ③ don't hold for W.

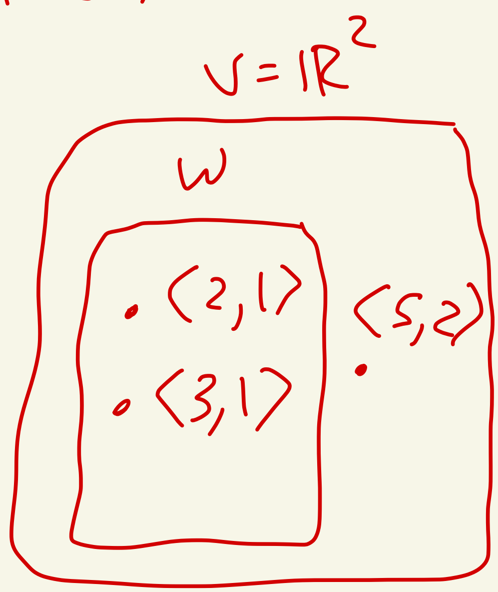
<u>For example:</u>

② Let $\vec{v}=\langle 2,1\rangle$ and $\vec{w}=\langle 3,1\rangle$. Then $\vec{v},\vec{w}$ are both in W.

However, $\vec{v}+\vec{w}=\langle 2,1\rangle+\langle 3,1\rangle$
$=\langle 5,2\rangle$

which isn't in W.

Thus, condition ② doesn't hold and W is not a subspace of $V=\mathbb{R}^2$.



Ex: Let $F = \mathbb{R}$ and

$$V = M_{2,2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 1 & 2 \\ 5 & \pi \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ldots \right\}$$

$\vec{0}$ in $M_{2,2}$

infinitely many more

We talked about how $M_{2,2}$ is vector space where vector addition is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

and scalar multiplication is given by

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

α in $F = \mathbb{R}$

Let

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{matrix} d = a+b \\ a,b,c,d \in \mathbb{R} \end{matrix} \right\}$$

$$= \left\{ \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}}_{2=1+1}, \underbrace{\begin{pmatrix} 5 & -10 \\ \frac{1}{2} & -5 \end{pmatrix}}_{-5=5-10}, \ldots \right\}$$

infinitely many more

Before we prove W is a subspace:

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$ because $0 = 0+0$.

$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 5 & -10 \\ \frac{1}{2} & -5 \end{pmatrix} \in W$ and $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 5 & -10 \\ \frac{1}{2} & -5 \end{pmatrix}$

$$= \begin{pmatrix} 6 & -9 \\ \frac{3}{2} & -3 \end{pmatrix} \in W$$

because $-3 = 6-9$

$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \in W$ and $3 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix} \in W$

because $6 = 3+3$

Let's prove that W is a subspace of $V = M_{2,2}$.

<u>proof:</u> We need to check the 3 criteria from the previous theorem.

① Is $\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in W?

Yes, if we set $a=b=c=d=0$ then $\vec{0} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\underbrace{d=a+b}_{0=0+0}$

② Is W closed under vector addition?

Let $\vec{v}$ and $\vec{w}$ be in W.

Then, $\vec{v} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

where $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in \mathbb{R}$

and $\underbrace{d_1 = a_1 + b_1}_{\text{since } \vec{v} \in W}$ and $\underbrace{d_2 = a_2 + b_2}_{\text{since } \vec{w} \in W}$

Then,

$$\vec{v}+\vec{w}=\begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}$$

Adding $d_1=a_1+b_1$ and $d_2=a_2+b_2$
gives $d_1+d_2 = a_1+b_1+a_2+b_2$

Regrouping gives

$$\boxed{d_1+d_2=(a_1+a_2)+(b_1+b_2)} \quad (*)$$

$(*)$ tells us that $\vec{v}+\vec{w}$ is in W.

So, W is closed under vector addition.

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(3) Let's show that W is closed under scalar multiplication.

Let $\vec{z} \in W$ and $\underbrace{\alpha \in \mathbb{R}}_{F=\mathbb{R}}$

Since $\vec{z} \in W$ we know that

$$\vec{z} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ where $a, b, c, d \in \mathbb{R}$ and $d = a + b$.

Then,

$$\alpha \vec{z} = \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

Multiplying $d = a + b$ by α gives

$$(\alpha d) = (\alpha a) + (\alpha b) \quad (**)$$

And $(**)$ tells us that

$$\alpha \vec{z} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$ is in W.

Thus, W is closed under scalar multiplication.

Since W satisfies properties
1, 2, and 3 above,
W is a subspace of $V = M_{2,2}$. $\blacksquare$